Unambiguous pure-state identification without classical knowledge

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We study how to unambiguously identify a given quantum pure state with one of the two reference pure states when no classical knowledge on the reference states is given but a certain number of copies of each reference quantum state are presented. By unambiguous identification, we mean that we are not allowed to make a mistake but our measurement can produce an inconclusive result. Assuming the two reference states are independently distributed over the whole pure state space in a unitary invariant way, we determine the optimal mean success probability for an arbitrary number of copies of the reference states and a general dimension of the state space. It is explicitly shown that the obtained optimal mean success probability asymptotically approaches that of the unambiguous discrimination as the number of the copies of the reference states increases.

I. INTRODUCTION

In quantum mechanics one cannot perfectly clone an unknown state [1], which makes the problem of distinguishing quantum states nontrivial [2, 3]. Imagine we are presented with an unknown quantum pure state ρ on a d-dimensional vector space $\boldsymbol{C}^d$. Let us assume that the input state ρ is guaranteed to be either one of two reference states ρ_1 and ρ_2, each being also a pure state on $\boldsymbol{C}^d$. Then how well can we identify the input state with one of the two reference states?

We can consider two cases depending on what kind of information on the reference states is available. In the first case, it is assumed that we have complete classical knowledge on the two reference states ρ_1 and ρ_2. This is the standard setting of quantum-state discrimination, which was solved by Helstrom [2].

On the other hand, we can also consider the case where only a certain number (N) of copies of ρ_1 and ρ_2 are presented, with no classical knowledge on them available [4]. See also related works in the case of qubits [5, 6]. In this case, we could obtain only limited classical information on the reference states, since the no-cloning theorem [1] does not allow us to increase the number of copies of the reference states. The best we can do is to perform a positive-operator-valued measure (POVM) measurement on the total state $\rho \otimes \rho_1^{\otimes N} \otimes \rho_2^{\otimes N}$ and try to identify the input state ρ with one of the reference states ρ_1 and ρ_2. If the number of copies, N, is infinite, the problem is reduced to quantum-state discrimination, since we could always obtain complete classical knowledge of a quantum state. In our previous paper [4], we called this problem "state identification" and determined the optimal mean identification probability for an arbitrary number (N) of copies of the reference states in a general dimension d.

In the standard setting of the discrimination problem, we are allowed to make an error and are interested in the optimal strategy that makes the error probability minimum. On the other hand, an error is not allowed in the problem of unambiguous discrimination [7, 8, 9]. Instead our measurement can produce one of three outputs 1, 2, or 0. If the output is 1(2), we are certain that the in-

put state ρ is $\rho_1(\rho_2)$, and the output 0 means that we do not know the identity of the input, which is called an inconclusive result. The optimal strategy is the one that minimizes the probability of the inconclusive result.

We can also generalize the unambiguous discrimination problem to the case in which a finite number (N) of copies of the reference states are presented without any classical information on them, which will be called the problem of unambiguous state identification in this paper. Bergou and Hillery studied this problem in the case of qubits ($d = 2$) when the number of copies of reference states, $N = 1$ [10]. They called the optimal strategy a programmable state discriminator since the strategy is not "hard wired" but supplied by the reference states stored in registers in the machine.

The problem of the quantum-state comparison [11, 12] is related to the state identification problem. One's task here is to establish whether or not two quantum systems have been prepared in the same state. The symmetry under interchanging the systems is essential in the state comparison, since the combined system is symmetric when the two systems are in the same state, whereas it has no definite exchange symmetry otherwise. The exchange symmetry plays a crucial role also in the state identification, but in a more involved way.

In this paper we study the problem of unambiguous identification of pure states for an arbitrary number N of copies of the reference states in a general dimension d. The two pure reference states ρ_1 and ρ_2 are independently distributed over the whole state space on $\boldsymbol{C}^d$ in a unitary invariant way. The input state ρ is assumed to be either one of the two reference states with the equal probabilities. We determine the optimal POVM and the optimal mean unambiguous identification probability, which can be explicitly shown to approach the mean unambiguous discrimination probability in the large-N limit.

II. MEAN UNAMBIGUOUS PURE STATE DISCRIMINATION

In this section we average the unambiguous discrimination probability, assuming that the two reference

states are independently distributed on $\boldsymbol{C}^d$ in a unitary-invariant way. This mean unambiguous discrimination probability will be later compared with the mean unambiguous identification probability in the large-N limit.

More precisely the unitary distribution of the reference states is specified in the following way. Expand a pure state as $|\phi\rangle = \sum_{i=1}^{d} c_i |i\rangle$ in terms of an orthonormal base $\{|i\rangle\}$ of $\boldsymbol{C}^d$. The distribution is then defined to be the one in which the $2d$-component real vector $\{x_i = \operatorname{Re} c_i, y_i = \operatorname{Im} c_i\}$ is uniformly distributed on the $(2d-1)$-dimensional hypersphere of radius 1 with the integration measure given by

$$dcdc^+ \equiv \prod_{i=1}^{d} (dx_i dy_i) \delta \left(\sum_i (x_i^2 + y_i^2) - 1 \right). \quad (1)$$

Evidently the distribution does not depend on the choice of the orthonormal base $\{|i\rangle\}$.

The optimal success probability of unambiguous discrimination of two known pure states $\rho_1 = |\phi_1\rangle\langle\phi_1|$ and $\rho_2 = |\phi_1\rangle\langle\phi_1|$ on $\boldsymbol{C}^d$ is given by [7, 8, 9]

$$p_{\max}(\rho_1, \rho_2) = 1 - |\langle \phi_1 | \phi_2 \rangle|. \quad (2)$$

We calculate the mean unambiguous discrimination probability

$$p_{\max}(d) = \left\langle p_{\max}(\rho_1, \rho_2) \right\rangle, \quad (3)$$

where $\langle \cdots \rangle$ means the average over ρ_1 and ρ_2, which are independently distributed according to the unitary distribution defined above.

The average $\langle |\langle \phi_1 | \phi_2 \rangle| \rangle$ can be calculated in terms of an integration over a $(2d-1)$-dimensional hypersphere as

$$\langle |\langle \phi_1 | \phi_2 \rangle| \rangle = \frac{\int dcdc^+ |c_1|}{\int dcdc^+ 1}, \quad (4)$$

with the integration measure given by Eq. (1).

We obtain

$$p_{\max}(d) = 1 - \frac{2^{d-1}(d-1)!}{(2d-1)!!}, \quad (5)$$

which is certainly less than the mean discrimination probability given in [4]:

$$\left\langle \frac{1}{2}(1 + \sqrt{1 - |\langle \phi_1 | \phi_2 \rangle|^2}) \right\rangle = \frac{1}{2} + \frac{d-1}{2d-1}. \quad (6)$$

III. UNAMBIGUOUS PURE STATE IDENTIFICATION

Suppose we are given an unknown pure state ρ on $\boldsymbol{C}^d$. We know that ρ is either one of the two reference states ρ_1 and ρ_2 on $\boldsymbol{C}^d$, with equal prior probabilities. Let us assume that we have no classical knowledge on the

reference states, but a certain number (N) of copies of each state are available. What is the optimal strategy to unambiguously identify the input state with one of the reference states when the two reference states are independently distributed over the whole pure-state space in a unitary invariant way? And what is the optimal mean probability of success?

We assume that the input state ρ is prepared in system 0 and N copies of each reference state ρ_a $(a = 1, 2)$ in systems $a_1, a_2, \ldots, a_N$, which will be collectively denoted by a. We specify the system which an operator acts on by the system number in the parentheses; namely, $\rho(0)$ means that this is an operator acting on system 0, for example.

Our task is then to unambiguously distinguish two states $\rho_1(0)\rho_1^{\otimes N}(1)\rho_2^{\otimes N}(2)$ and $\rho_2(0)\rho_1^{\otimes N}(1)\rho_2^{\otimes N}(2)$. The mean success probability of identification is given by

$$p^{(N)}(d) = \frac{1}{2} \sum_{a=1}^{2} \left\langle \operatorname{tr} \left[E_a \rho_a(0)\rho_1^{\otimes N}(1)\rho_2^{\otimes N}(2) \right] \right\rangle, \quad (7)$$

where $\{E_0, E_1, E_2\}$ is a POVM acting on the whole system and $\langle \cdots \rangle$ represents the average over ρ_1 and ρ_2 defined in the preceding section. When the outcome of the POVM is a $(= 1, 2)$, we identify the input ρ with ρ_a with certainty. Outcome 0 of the POVM means we have an inconclusive result. Note that the POVM should be independent of ρ_1 and ρ_2 since we are given no classical knowledge on them.

The average over the reference states can be easily performed by the use of formula for the average of the n-fold tensor product of an identical pure state ρ

$$\langle \rho^{\otimes n} \rangle = \frac{\mathcal{S}_n}{d_n}, \quad (8)$$

where $\mathcal{S}_n$ is the projection operator onto the totally symmetric subspace and d_n is its dimension, $d_n = \operatorname{tr}[\mathcal{S}_n] = {}_{n+d-1}C_{d-1}$ [13]. We find

$$p^{(N)}(d) = \frac{1}{2d_{N+1}d_N} \Big(\operatorname{tr}[E_1 \mathcal{S}_{N+1}(01)\mathcal{S}_N(2)]$$
$$+ \operatorname{tr}[E_2 \mathcal{S}_N(1)\mathcal{S}_{N+1}(02)] \Big), \quad (9)$$

where $\mathcal{S}_{N+1}(01)$ is the projector onto the totally symmetric subspace on systems $(0, 1) = (0, 1_1, 1_2, \ldots, 1_N)$ and other $\mathcal{S}$'s are defined similarly.

The POVM should satisfy the following conditions:

$$E_0, E_1, E_2 \geq 0, \quad E_0 + E_1 + E_2 = 1 \quad (10)$$

and, for any ρ_1 and ρ_2,

$$\operatorname{tr}\left[E_1 \rho_2(0)\rho_1^{\otimes N}(1)\rho_2^{\otimes N}(2)\right] = 0,$$
$$\operatorname{tr}\left[E_2 \rho_1(0)\rho_1^{\otimes N}(1)\rho_2^{\otimes N}(2)\right] = 0, \quad (11)$$

which implies no error is allowed. It is evident that the above no-error conditions are equivalent to

$$E_1 \mathcal{S}_N(1)\mathcal{S}_{N+1}(02) = \mathcal{S}_N(1)\mathcal{S}_{N+1}(02)E_1 = 0,$$
$$E_2 \mathcal{S}_N(2)\mathcal{S}_{N+1}(01) = \mathcal{S}_N(2)\mathcal{S}_{N+1}(01)E_2 = 0. \quad (12)$$

Now we observe that the set of POVM's satisfying conditions (10) and (12) is convex. Namely, if each of two POVM's E_a and E'_a respects conditions (10) and (12), so does their convex linear combination $qE_a + (1-q)E'_a$ for any $0 \leq q \leq 1$. And the resulting probability, Eq.(9), is also a convex combination: $p(qE + (1-q)E') = qp(E) + (1-q)p(E')$ in an obvious abbreviated notation.

We exploit this convexity of the POVM to impose some symmetries on the optimal POVM without loss of generality. First we notice the problem is symmetric under the exchange between systems 1 and 2. Suppose a POVM F_a is optimal. Then another POVM F'_a, defined by

$$ F'_1 = TF_2T, \ F'_2 = TF_1T, \ F'_0 = TF_0T, \qquad (13) $$

is also legitimate and optimal. Here we introduced the exchange operator T between systems 1 and 2. Then a new POVM $E_a = \frac{1}{2}(F_a + F'_a)$ is also optimal and satisfies the exchange symmetry between systems 1 and 2,

$$ E_2 = TE_1T, \ E_0 = TE_0T. \qquad (14) $$

The second symmetry we consider is the unitary symmetry of the distribution of the reference states. If a POVM F_a is optimal, another POVM defined by

$$ U^{\otimes(2N+1)}F_a(U^+)^{\otimes(2N+1)} \ (a = 0, 1, 2) \qquad (15) $$

is also legitimate and optimal for any unitary operator U. Let us construct a POVM by

$$ E_a = \int dU U^{\otimes(2N+1)}F_a(U^+)^{\otimes(2N+1)} $$
$$ (a = 0, 1, 2), \qquad (16) $$

where dU is the normalized positive-invariant measure of the group $U(d)$. The new POVM E_a is clearly a legitimate optimal POVM. Furthermore, since E_a commutes with $U^{\otimes(2N+1)}$ for any U, we conclude that E_a is a scalar with respect to the group $U(d)$. Thus we can assume that the optimal POVM satisfies the exchange symmetry of Eqs. (14) and is scalar with respect to the group $U(d)$.

By the exchange symmetry, the mean probability, Eq.(9), to be optimized takes the form

$$ p^{(N)}(d) = \frac{1}{d_{N+1}d_N} \mathrm{tr}\left[E_1 \mathcal{S}_{N+1}(01)\mathcal{S}_N(2)\right]. \qquad (17) $$

And the conditions E_1 should satisfy are given by

$$ E_1 \geq 0, \ 1 \geq E_1 + TE_1T, \qquad (18) $$

and the no-error conditions

$$ E_1\mathcal{S}_N(1)\mathcal{S}_{N+1}(02) = \mathcal{S}_N(1)\mathcal{S}_{N+1}(02)E_1 = 0. \qquad (19) $$

Finally we note that we can work in the subspace V_{sym}, in which each of systems 1 and 2 is both totally symmetric. With this in mind, we set $\mathcal{S}_N(1) = \mathcal{S}_N(2) = 1$ hereafter.

IV. CASE OF QUBITS ($d = 2$)

In this section we study the case of qubits ($d = 2$), where the individual system can be regarded as a spin-1/2 particle and the problem reduces to the angular momentum recoupling. In the subspace V_{sym}, each system $a \ (=1, 2)$ consisting of N spin-1/2 particles is totally symmetric, implying the total angular momentum of each system is $j \equiv N/2$.

We can construct the total angular momentum of the whole $2N + 1$ systems in two ways. First the combined system of 0 and 1 has the angular momentum $J_1 = j_- \equiv j - 1/2$ or $J_1 = j_+ \equiv j + 1/2$. Then this intermediate angular momentum J_1 is coupled with the angular momentum j of system 2, resulting in the total angular momentum of the whole system J. Using the standard notation [14], we write the resultant eigenstate with the total angular momentum J and its z component M as

$$ |A_{J_1}; JM\rangle \equiv |(j\frac{1}{2})J_1, j; JM\rangle \ (J_1 = j_-, j_+), \quad (20) $$

where we ordered three systems as $1 \otimes 0 \otimes 2$ on the right-hand side. Note that the state $|A_{j_+}; JM\rangle$ is totally symmetric in the subspace of systems 0 and 1 and the state $|A_{j_-}; JM\rangle$ is not, that is,

$$ \mathcal{S}_{N+1}(01)|A_{j_+}; JM\rangle = |A_{j_+}; JM\rangle, $$
$$ \mathcal{S}_{N+1}(01)|A_{j_-}; JM\rangle = 0. \qquad (21) $$

Another coupling scheme is that systems 0 and 2 are first coupled to the intermediate angular momentum J_2. This coupling scheme defines another orthonormal base in the whole space,

$$ |B_{J_2}; JM\rangle \equiv |j, (\frac{1}{2}j)J_2; JM\rangle \ (J_2 = j_+, j_-), \quad (22) $$

where the three systems are ordered in the same way as in Eq. (20). The state $|B_{J_2}; JM\rangle$ has the following exchange symmetries:

$$ \mathcal{S}_{N+1}(02)|B_{j_+}; JM\rangle = |B_{j_+}; JM\rangle, $$
$$ \mathcal{S}_{N+1}(02)|B_{j_-}; JM\rangle = 0. \qquad (23) $$

For a given set of $J (\neq 2j + 1/2)$ and M, the two bases (20) and (22) are related by a unitary matrix, which can be taken to be real by the standard phase convention,

$$ |A_{J_1}; JM\rangle = \sum_{J_2=j_+, j_-} R^J_{J_1 J_2} |B_{J_2}; JM\rangle, \qquad (24) $$

where the recoupling coefficient $R^J_{J_1 J_2}$ is expressed by the Racah coefficient,

$$ R^J_{J_1 J_2} = \sqrt{(2J_1 + 1)(2J_2 + 1)} \, W(j\frac{1}{2}Jj; J_1 J_2), \quad (25) $$

and its explicit form is given by the following 2×2 orthogonal matrix:

$$R^J = \begin{pmatrix} \frac{J+\frac{1}{2}}{2j+1} & \frac{\sqrt{(2j+J+\frac{3}{2})(2j-J+\frac{1}{2})}}{2j+1} \\ \frac{\sqrt{(2j+J+\frac{3}{2})(2j-J+\frac{1}{2})}}{2j+1} & -\frac{J+\frac{1}{2}}{2j+1} \end{pmatrix}, \tag{26}$$

where rows and columns are allocated in the descending order of J_1 and J_2, respectively [14].

Now the no-error conditions (19) imply that E_1 is an operator in the space spanned by $|B_{j_-}; JM\rangle$, which is annihilated by $\mathcal{S}_{N+1}(02)$ as shown in Eq. (23). Furthermore, E_1 can be assumed to be a $U(2)$ scalar owing to the argument in the preceding section. This means that E_1 is diagonal with respect to J and is proportional to the identity for M. Combining these two properties, we find that E_1 should have the form

$$E_1 = \sum_{J=\frac{1}{2}}^{2j-\frac{1}{2}} e_J \sum_{M=-J}^{J} |B_{j_-}; JM\rangle\langle B_{j_-}; JM|, \tag{27}$$

where coefficients e_J should be non-negative by the positivity of E_1.

An upper bound is further imposed on the coefficient e_J by the remaining condition $1 \geq E_1 + T E_1 T$ in Eq.(18). This condition can be written as

$$1 \geq \sum_{J=\frac{1}{2}}^{2j-\frac{1}{2}} e_J \sum_{M=-J}^{J} \Big(|B_{j_-}; JM\rangle\langle B_{j_-}; JM| $$
$$+ |A_{j_-}; JM\rangle\langle A_{j_-}; JM| \Big)$$
$$= \sum_{J=\frac{1}{2}}^{2j-\frac{1}{2}} e_J \sum_{M=-J}^{J} \sum_{J_1, J_2} |B_{J_1}; JM\rangle O_{J_1 J_2}^{(J)}\langle B_{J_2}; JM|, \tag{28}$$

where the matrix $O_{J_1 J_2}^{(J)}$ is expressed in terms of the recoupling coefficients as follows:

$$O^{(J)} = \begin{pmatrix} \left(R_{j_- j_+}^J\right)^2 & R_{j_- j_+}^J R_{j_- j_-}^J \\ R_{j_- j_+}^J R_{j_- j_-}^J & 1 + \left(R_{j_- j_-}^J\right)^2 \end{pmatrix}. \tag{29}$$

Eigenvalues of $O^{(J)}$ are readily calculated and found to be $1 \pm |R_{j_- j_-}^J|$. Thus constraints on the coefficient e_J are given by

$$0 \leq e_J \leq \frac{1}{1 + |R_{j_- j_-}^J|}. \tag{30}$$

Now it is easy to express the trace in Eq.(17) in terms of the recoupling coefficients and e_J:

$$\mathrm{tr}\,[E_1 \mathcal{S}_{N+1}(01)\mathcal{S}_N(2)]$$

$$= \sum_{JM} \langle A_{j_+}; JM | E_1 | A_{j_+}; JM\rangle$$
$$= \sum_{JM} \left(R_{j_+ j_-}^J\right)^2 \langle B_{j_-}; JM | E_1 | B_{j_-}; JM\rangle$$
$$= \sum_{J=\frac{1}{2}}^{2j-\frac{1}{2}} (2J+1) \left(R_{j_+ j_-}^J\right)^2 e_J. \tag{31}$$

Therefore the probability, Eq.(17), reaches its maximum when the coefficients e_J takes its upper bound given in Eq.(30).

Thus the optimal mean unambiguous identification probability is given by

$$p_{\max}^{(N)}(d=2) = \frac{2J+1}{2_{N+1} 2_N} \sum_{J=\frac{1}{2}}^{2j-\frac{1}{2}} \frac{\left(R_{j_+ j_-}^J\right)^2}{1 + |R_{j_- j_-}^J|}$$

$$= \frac{2J+1}{2_{N+1} 2_N} \sum_{J=\frac{1}{2}}^{2j-\frac{1}{2}} \left(1 - |R_{j_- j_-}^J|\right). \tag{32}$$

We used the orthogonality of the recoupling matrix in the above derivation. Inserting the explicit form of the recoupling coefficients and performing the sum in the above expression, we find a simple formula for $p_{\max}^{(N)}(d=2)$:

$$p_{\max}^{(N)}(d=2) = \frac{N}{3(N+1)}. \tag{33}$$

The optimal POVM is then given by

$$E_1 = \sum_{J=\frac{1}{2}}^{2j-\frac{1}{2}} \frac{1}{1 + |R_{j_- j_-}^J|} \sum_{M=-J}^{J} |B_{j_-}; JM\rangle\langle B_{j_-}; JM|,$$

$$E_2 = T E_1 T, \quad E_0 = 1 - E_1 - E_2. \tag{34}$$

As N goes to infinity, $p_{\max}^{(N)}(d=2)$ approaches $1/3$, which is equal to the mean unambiguous discrimination probability $p_{\max}(d=2)$ given in Eq. (5). When $N=1$, on the other hand, the optimal POVM takes the form:

$$E_1 = \frac{2}{3}(1 - \mathcal{S}_2(02)), \quad E_2 = \frac{2}{3}(1 - \mathcal{S}_2(01)),$$
$$E_0 = 1 - E_1 - E_2. \tag{35}$$

which reproduces the one given by Bergou and Hillery [10] in the case of the equal prior probabilities. For a two-spin-1/2-particle system, the state is either symmetric (triplet state) or antisymmetric (singlet state). Therefore, the optimal POVM for $N=1$ can also be written as

$$E_1 = \frac{2}{3}|\Phi(02)\rangle\langle\Phi(02)|, \quad E_2 = \frac{2}{3}|\Phi(01)\rangle\langle\Phi(01)|, \tag{36}$$

where we introduced the singlet state $|\Phi(01)\rangle = (|0\rangle|1\rangle - |1\rangle|0\rangle)/\sqrt{2}$ for systems 0 and 1 and similarly $|\Phi(02)\rangle$ for systems 0 and 2.

One might wonder if we really need the complicated explicit form (26) of the Racah coefficients to obtain the simple final result of Eq. (33). Actually we can avoid the explicit use of Racah coefficients if we exploit the algebraic properties of angular momentum operators. We will show it in the general dimensional case treated in the next section.

V. CASE OF ARBITRARY DIMENSION d

In this section, we generalize the argument in the preceding section to the arbitrary dimensional case. The essential point was the intimate relation between the symmetry properties under system permutations and the angular momentum of the combined system. The symmetry under system permutations is characterized by the representation of the symmetric group S_{2N+1}. And the angular momentum specifies the representation of $SU(2)$, more generally the unitary group $U(2)$. Therefore, in the case of arbitrary dimension d, we should classify the states according to representations of the symmetric group S_{2N+1} and the unitary group $U(d)$.

Let us introduce the orthonormal base of the total space $(\boldsymbol{C}^d)^{\otimes(2N+1)}$ according to irreducible representations of the symmetric group S_{2N+1} and the unitary group $U(d)$. We write states in this base as

$$| \lambda, a, b \rangle. \tag{37}$$

Here λ represents an irreducible representation of S_{2N+1}, which is specified by a Young diagram. By the expression $\lambda = [\lambda_1, \lambda_2, \ldots]$, we denote a Young diagram consisting of a set of rows with their lengths given by $\lambda_1, \lambda_2, \ldots$. The label a indexes orthogonal vectors in a particular S_{2N+1} representation space and it runs from 1 to the dimension of the S_{2N+1} representation. It is known that the λ also specifies irreducible representations of the unitary group $U(d)$ and its vectors are indexed by b, which runs from 1 to $m_\lambda(d)$, the multiplicity of representation λ of S_{2N+1} on $(\boldsymbol{C}^d)^{\otimes(2N+1)}$ [15].

As stated before, we can work in the subspace V_{sym}, where systems 1 and 2 are both totally symmetric, $\mathcal{S}_N(1) = 1$ and $\mathcal{S}_N(2) = 1$. Possible Young diagrams λ appearing in V_{sym} and the range of the index a associated with a particular λ can be determined by decomposing the product of three $U(d)$ irreducible representations $[1] \otimes [N] \otimes [N]$. We decompose the space V_{sym} into three orthogonal subspaces V_n ($n = 1, 2, 3$) according to the number of rows, n, of the Young's diagram (see Fig. 1).

The subspace V_1 consists of totally symmetric states:

$$| [2N+1], b \rangle, \quad b = 1, \ldots, m_{[2N+1]}(d), \tag{38}$$

where we omitted the index a, since the totally symmetric representation of S_{2N+1} is one dimensional. The states in V_2 belong to representations specified by Young's diagrams of two rows $[\lambda_1, \lambda_2]$, where $N + 1 \leq \lambda_1 \leq 2N$ and
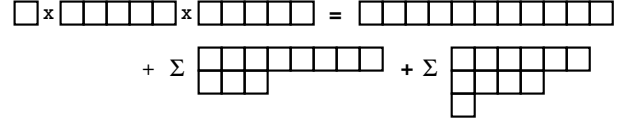

FIG. 1: Decomposition of the product of three $U(d)$ irreducible representations $[1] \otimes [N] \otimes [N]$. The decomposition leads to the three orthogonal subspaces V_n ($n = 1, 2, 3$) according to the number of rows, n, of the Young's diagram.

$\lambda_2 = 2N + 1 - \lambda_1$. Since each of these $U(d)$ representations appears twice in V_{sym}, we distinguish the two by label $a = 1, 2$ as follows:

$$| [\lambda_1, \lambda_2], a, b \rangle, \quad a = 1, 2, \quad b = 1, \ldots, m_{[\lambda_1, \lambda_2]}(d). \tag{39}$$

The remaining states are those whose Young's diagram has three rows and span the subspace V_3.

$$| [\lambda_1, \lambda_2, 1], b \rangle, \quad b = 1, \ldots, m_{[\lambda_1, \lambda_2, 1]}(d), \tag{40}$$

where $N \leq \lambda_1 \leq 2N - 1$ and $\lambda_2 = 2N - \lambda_1$. We do not need the label a for these states, because each representation of this type occurs only once in V_{sym}. Note that the length of the third row is always 1.

Now let us determine a possible form of the POVM elements E_1 and E_2. First of all, E_1 should respect the no-error conditions, Eq. (19). If $| x \rangle$ is in V_1, it is clear that $\mathcal{S}_{N+1}(02)| x \rangle = | x \rangle$. It is also easy to see that $\mathcal{S}_{N+1}(02)| x \rangle = 0$ for $| x \rangle \in V_3$, because representations with the Young's diagram of three rows cannot be constructed otherwise. States in V_2 for a given set of λ and b can be constructed in two different ways. We can assume the label $a = 1, 2$ for states in V_2 is chosen such that

$$\begin{aligned} \mathcal{S}_{N+1}(02)| [\lambda_1, \lambda_2], 1, b \rangle &= 0, \\ \mathcal{S}_{N+1}(02)| [\lambda_1, \lambda_2], 2, b \rangle &= | [\lambda_1, \lambda_2], 2, b \rangle. \end{aligned} \tag{41}$$

We should also remember that the POVM element E_1 can be chosen to be a scalar with respect to $U(d)$. All these facts lead to the following form for E_1:

$$E_1 = \sum_{\lambda_1} e_{[\lambda_1, \lambda_2]} \sum_b | [\lambda_1, \lambda_2], 1, b \rangle \langle [\lambda_1, \lambda_2], 1, b |$$
$$+ \sum_{\lambda_1} e_{[\lambda_1, \lambda_2, 1]} \sum_b | [\lambda_1, \lambda_2, 1], b \rangle \langle [\lambda_1, \lambda_2, 1], b |, \tag{42}$$

where e_λ are some non-negative coefficients. The coefficient e_λ depends only on λ, and the orthonormal states $| [\lambda_1, \lambda_2], 1, b \rangle$ and $| [\lambda_1, \lambda_2, 1], b \rangle$ are complete in the space annihilated by the projector $\mathcal{S}_{N+1}(02)$. Therefore, this E_1 can be most conveniently expressed as the following operator form:

$$\begin{aligned} E_1 &= e\Big(1 - \mathcal{S}_{N+1}(02)\Big), \\ e &= \sum_\lambda e_\lambda \Gamma_\lambda, \end{aligned} \tag{43}$$

where Γ_λ is the projection operator onto the $U(d)$ representation space specified by λ. Furthermore, we can express E_2 as

$$E_2 = e\Big(1 - \mathcal{S}_{N+1}(01)\Big), \qquad (44)$$

by the same operator e, since Γ_λ is symmetric under the exchange of systems 1 and 2 and we can assume $E_2 = TE_1T$ owing to the conclusion in Sec. II.

Now that we have determined the possible form of E_1 and E_2, we can proceed to the positivity condition of E_0: namely, $1 \geq E_1 + E_2$. This condition can be written as

$$\begin{aligned} 1 \geq E_1 + E_2 &= e(2 - \mathcal{S}_{N+1}(01) - \mathcal{S}_{N+1}(02)) \\ &= e(1-A), \qquad (45) \end{aligned}$$

where we introduced an operator A in the subspace V_{sym} to be

$$A \equiv \mathcal{S}_{N+1}(01) + \mathcal{S}_{N+1}(02) - 1. \qquad (46)$$

It is convenient to introduce another operator, which is the difference of the two projectors:

$$D \equiv \mathcal{S}_{N+1}(01) - \mathcal{S}_{N+1}(02). \qquad (47)$$

Note that operators A and D are diagonal with respect to λ and proportional to identity for the index b, since these operators involve just permutation operators. We also observe the relations

$$A^2 = 1 - D^2, \qquad (48)$$
$$AD + DA = 0, \qquad (49)$$

which can be shown by an explicit calculation using $\mathcal{S}_{N+1}(01)^2 = \mathcal{S}_{N+1}(01)$ and $\mathcal{S}_{N+1}(02)^2 = \mathcal{S}_{N+1}(02)$.

The operator A is -1 in the subspace V_3, since both projectors $\mathcal{S}_{N+1}(01)$ and $\mathcal{S}_{N+1}(02)$ annihilate any states in V_3. In the subspace V_2, two eigenvalues of A have opposite signs in the invariant subspace associated with a given set of λ and b. This is because A and D anticommute and the operation of D changes the sign of eigenvalue of A. Note that D does not annihilate any state in V_2. Combining these facts, we conclude that the positivity condition of (45) implies the following inequality:

$$\frac{1}{1+|A|} \geq e, \qquad (50)$$

in subspaces V_2 and V_3.

Let us go back to the mean success probability, Eq.(17), and write it with the form of E_1 given by Eq.(43).

$$p^{(N)}(d) = \frac{1}{d_{N+1}d_N}\text{tr}\left[e(1 - \mathcal{S}_{N+1}(02))\mathcal{S}_{N+1}(01))\right], (51)$$

which can be further rewritten as

$$p^{(N)}(d) = \frac{1}{2d_{N+1}d_N}\text{tr}\left[e(1 - A^2)\right]. \qquad (52)$$

In the above equation, we find that the subspaces V_1 and V_2 have no contribution to the trace sum. And $|A|$ in the upper bound of e in Eq.(50) commutes with $1 - A^2$ in the trace. Therefore, we immediately obtain the optimal mean success probability as follows:

$$\begin{aligned} p^{(N)}(d) &\leq \frac{1}{2d_{N+1}d_N}\text{tr}\left[\frac{1}{1+|A|}(1-A^2)\right] \\ &= \frac{1}{2d_{N+1}d_N}\text{tr}\left[1 - |A|\right] \\ &\equiv p_{\max}^{(N)}(d). \qquad (53) \end{aligned}$$

The optimal success probability is thus attained by

$$\begin{aligned} E_1 &= \frac{1}{1+|A|}(1 - \mathcal{S}_{N+1}(02)), \\ E_2 &= \frac{1}{1+|A|}(1 - \mathcal{S}_{N+1}(01)), \\ E_0 &= \frac{A+|A|}{1+|A|}. \qquad (54) \end{aligned}$$

Here we took $e_{[\lambda_1,\lambda_2,1]} = \frac{1}{2}$ for simplicity, which is the maximum value allowed by Eq.(50), though the subspace V_3 does not contribute to $p_{\max}^{(N)}(d)$.

We must still determine eigenvalues of $|A|$ in the subspace V_2 in order to evaluate $p_{\max}^{(N)}(d)$ further. As mentioned above, the operator A is proportional to identity with respect to the index b for a given particular λ. Therefore, eigenvalues of A are independent of the dimension d up to multiplicity. Thus we can assume the dimension d is equal to 2, which allows us to exploit the angular momentum algebra.

Assuming $d = 2$, we introduce three sets of angular momentum operators: $\boldsymbol{s}(0) \equiv \frac{1}{2}\boldsymbol{\sigma}(0)$ for system 0, $\boldsymbol{j}(1)$ for system 1, and $\boldsymbol{j}(2)$ for system 2. In the subspace V_{sym}, we have $\boldsymbol{j}(a)^2 = \frac{N}{2}(\frac{N}{2}+1)$, since the total angular momentum of system $a = 1,2$ is $\frac{N}{2}$. It is then easy to show that the projector $\mathcal{S}_{N+1}(0a)$ can be written in terms of the angular momentum operators as follows ($a = 1,2$):

$$\mathcal{S}_{N+1}(0a) = \frac{1}{N+1}\left(2\boldsymbol{j}(a)\cdot\boldsymbol{s}(0) + \frac{N}{2} + 1\right). \quad (55)$$

Using this form for the projectors, we calculate A^2. After some algebraic calculation involving the Pauli matrices and angular momentum commutation relations, we find

$$A^2 = \frac{1}{(N+1)^2}\left(\boldsymbol{J}^2 + \frac{1}{4}\right), \qquad (56)$$

where $\boldsymbol{J} = \boldsymbol{s}(0) + \boldsymbol{j}(1) + \boldsymbol{j}(1)$ is the total angular momentum operator. The eigenvalue of $\boldsymbol{J}^2$ is $J(J+1)$ ($J = \frac{1}{2},\ldots,N+\frac{1}{2}$), which implies that eigenvalues of A are given by $\pm\frac{J+\frac{1}{2}}{N+1}$ with multiplicity $2J+1$.

For a general dimension d, we thus conclude that A in V_2 has eigenvalues $\pm\frac{\lambda_1-N}{N+1}$ with multiplicity $m_{[\lambda_1,\lambda_2]}(d)$, since the total angular momentum J in the case of $d = 2$ is given by $J = \frac{\lambda_1-\lambda_2}{2} = \lambda_1 - N - \frac{1}{2}$.
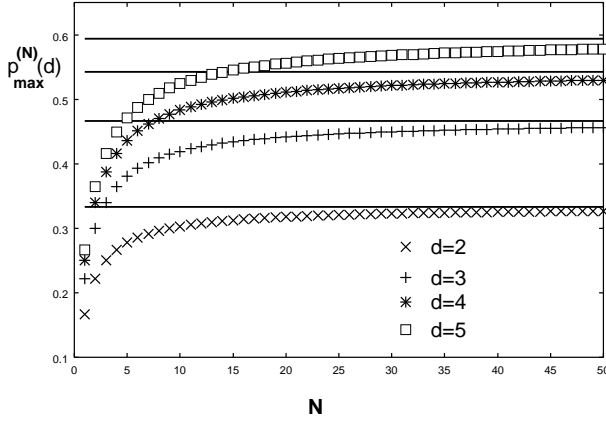
FIG. 2: The optimal mean unambiguous identification probability $p_{\max}^{(N)}(d)$ as a function of the number of the copies (N) of the reference states. As N increases, $p_{\max}^{(N)}(d)$ approaches the mean optimal unambiguous discrimination probability shown by the horizontal lines.

Finally we obtain the formula for the optimal success probability

$$
p_{\max}^{(N)}(d) = \frac{1}{d_{N+1}d_N} \times \\
\sum_{\lambda_1=N+1}^{2N} m_{[\lambda_1\lambda_2]}(d)\left(1-\frac{\lambda_1-N}{N+1}\right), \quad (57)
$$

where $m_{[\lambda_1,\lambda_2]}(d)$ ($\lambda_2 = 2N+1-\lambda_1$) is the multiplicity of the S_{2N+1} irreducible representation $[\lambda_1,\lambda_2]$ and given by [15]

$$
m_{[\lambda_1,\lambda_2]}(d) = \frac{(\lambda_1+d-1)!(\lambda_2+d-2)!(\lambda_1-\lambda_2+1)}{(d-1)!(d-2)!(\lambda_1+1)!\lambda_2!}. \quad (58)
$$

Let us study the asymptotic value of $p_{\max}^{(N)}(d)$ when the number of the copies N is very large. In this case we can replace the sum in Eq.(57) by a continuous integration with respect to $x = \frac{\lambda_1}{N}-1$. We find

$$
p_{\max}^{(N)}(d) \to 2(d-1)\int_0^1 dx\,(1+x)^{d-2}(1-x)^{d-1} \\
= 1 - \frac{2^{d-1}(d-1)!}{(2d-1)!!} \quad (N\to\infty), \quad (59)
$$

which is equal to $p_{\max}(d)$ given by Eq. (5). Thus, as expected, the unambiguous identification reduces to the unambiguous discrimination as the number of the copies goes to infinity. Figure 2 displays how the unambiguous identification probability approaches the unambiguous discrimination probability as the number of the copies increases.

VI. CONCLUDING REMARKS

We have studied the problem of unambiguously identifying the input state of a d-dimensional system with one of the two reference states when N copies of each reference state are presented with no classical information. We have determined the optimal mean unambiguous identification probability $p_{\max}^{(N)}(d)$ as a function of d and N.

It is interesting to compare the results in this paper and those for the identification problem without the no-error conditions, which was studied in our previous paper [4]. In both problems the symmetry under system permutations plays an essential role. This is also true in the state comparison studied by Barnett, Chefles, and Jex [11], in which one's task is to establish whether or not two quantum systems have been prepared in the same state. In this case the symmetry under exchanging the two systems can characterize the optimal POVM. In the state identification, however, we must distinguish the two states $\rho_1(0)\rho_1^{\otimes N}(1)\rho_2^{\otimes N}(2)$ and $\rho_2(0)\rho_1^{\otimes N}(1)\rho_2^{\otimes N}(2)$. Therefore, we must consider the symmetries with respect to partial permutations among systems 0 and 1 and among systems 0 and 2. The relevant operators are noncommutable projection operators $\mathcal{S}_{N+1}(01)$ and $\mathcal{S}_{N+1}(02)$, which makes the optimization of the success probability rather involved. The success probability is expressed by the trace of the modulus of some linear combination of the symmetrizers $\mathcal{S}_{N+1}(01)$ and $\mathcal{S}_{N+1}(02)$: $D = \mathcal{S}_{N+1}(01) - \mathcal{S}_{N+1}(02)$ in the case of the identification problem without the no-error conditions and $A = \mathcal{S}_{N+1}(01) + \mathcal{S}_{N+1}(02) - 1$ in the unambiguous identification problem considered in this paper.

As for the optimal POVM, it was shown that the optimal success probability can be attained by a projective measurement in the identification problem without the no-error conditions, whereas the optimal POVM obtained for the unambiguous identification considered here is not a projective measurement.

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